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Heat conduction in a semi-infinite solid due to time-dependent laser source

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Abstract—An analytical model for the computation of temperature and heat flux distribution in a semi-infinite solid when subjected to spatially decaying, time-dependent laser source is investigated. The appropriate dimensionless parameters are identified and the reduced temperature and heat flux as a function of these parameters are presented in a graphic form. Some special cases of practical interest are also discussed. It is demonstrated that the present analysis covers the continuously operating constant strength as well as instantaneous laser source cases, along with some new solutions. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

There is an increasing interest in material processing using lasers; particularly, laser drilling, machining, and welding have been studied analytically and experimentally. Several such applications are discussed in Rykalin *et al.* [1] and Ready [2]. Dabby and Paek [3] observed several thermally induced effects when an intense laser radiation is incident upon a heat-transfer surface. One of these effects is the 'explosive removal of material'. A possible explanation for this phenomenon given by Blackwell [4] is that the point of maximum temperature (before the phase change occurs at the exposed surface) lies inside the body because of the heat loss to the surroundings. We note that for a material which expands on changing phase and the initial phase change occurs inside of the body instead of at the exposed surface, then the explosive material removal is expected.

Blackwell [4] investigated this material removal phenomenon analytically by calculating the temperature profile in a semi-infinite body with an exponentially decaying (with position) source and convective boundary condition. He showed that the location of the maximum temperature is a strong function of the dimensionless parameters such as Biot and Fourier numbers. Zubair and Chaudhry [5] discussed the fundamental solution to the problem considered by Blackwell [4]. They provided an analytic solution to the problem in which the material is subjected to an instantaneous, exponentially decaying (with position) laser source. The objective of this paper is to discuss an analytical solution of a semi-infinite solid due to a general time-dependent laser source and convective-boundary condition.

2. MATHEMATICAL FORMULATION

The heat conduction equation describing the temperature distribution in a semi-infinite, homogeneous and isotropic body with an energy source term is given by [6–8]

$$\rho C_{\mathrm{p}} \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + q'''. \tag{1}$$

We consider a general time-dependent exposure of laser radiation which is absorbed within the material and has the effect of an internally distributed heat source. This is typically true for organic materials [2], in which the absorption coefficient is considerably smaller and the energy is deposited over a greater thickness. Thus, the energy source term in equation (1) may be modeled as [4, 5]

$$q'''(x,t) = \dot{I}_0(t)\mu(1-R)\,\mathrm{e}^{-\mu x} \tag{2}$$

where $\dot{I}_0(t)$ is the time-dependent radiation intensity, R is the surface reflectance and μ is the material absorption coefficient. This model assumes no spatial variation of $\dot{I}_0(t)$ in the plane normal to the beam. Also, the problem times are sufficiently small so that the diffusion perpendicular to the beam (x) can be ignored [2].

The appropriate initial and boundary conditions are assumed to be the following:

$$T(x,0) = T_{\mathrm{int}} \tag{3}$$

$$-k \frac{\partial T(0,t)}{\partial x} = h[T_{\infty} - T(0,t)] \tag{4}$$

and

NOMENCLATURE

Bi	Biot number ($Bi = hx/k$)
Bo	dimensionless energy absorption $[Bo = \dot{I}_0\mu(1-R)/2\rho C_p(\lambda^2-\alpha\mu^2) \times (T_\infty - T_{int})]$
C_p	specific heat at constant pressure [J kg^{-1} K^{-1}]
Fo	Fourier number ($Fo = \alpha t/x^2$)
h	convective heat transfer coefficient [W m^{-2} K^{-1}]
I_0	energy released by laser source [J m^{-2}]
$\dot{I}_0$	rate of energy released by laser source [W m^{-2}]
k	thermal conductivity [W m^{-1} K^{-1}]
Q	dimensionless heat flux
q''	heat flux [W m^{-2}]
q'''	rate of energy generation per unit volume [W m^{-3}]
R	surface reflectance
s	Laplace transform variable
t	time [s]
T	temperature [K]
x	spatial variable [m].

Greek symbols

α	thermal diffusivity ($\alpha = k/\rho C_p$) [m^2 s^{-1}]
Γ	Gamma function
τ	reduced time constant ($\tau = \lambda x/\sqrt{\alpha}$)
λ	laser source constant [s$^{-1/2}$]
χ	dimensionless distance ($\chi = \mu x$)
θ	temperature, ($\theta = T - T_{int}$) [K]
Θ	dimensionless temperature
ρ	density [kg m^{-3}]
μ	absorption coefficient [1/m].

Subscripts

int	initial
∞	free-stream
1	instantaneous
2	exponential-type
3	constant surface temperature case
21	at the wall
22	constant strength
23	no heat generation case.

$$\frac{\partial T(\infty, t)}{\partial x} = 0. \tag{5}$$

It should be noted that Blackwell [4] used the above boundary and initial conditions to explain the explosive removal phenomenon in a semi-infinite solid due to a continuously operating, constant strength laser source.

Defining θ as the temperature rise above the initial temperature

$$\theta = (T - T_{int}) \tag{6}$$

and substituting in the above equations, we get

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} + \frac{(1-R)}{\rho C_p} \dot{I}_0(t)\mu\, e^{-\mu x} \tag{7}$$

$$\theta(x, 0) = 0 \tag{8}$$

$$\frac{\partial \theta(0,t)}{\partial x} + \frac{h}{k}[(T_\infty - T_{int}) - \theta(0,t)] = 0 \tag{9}$$

$$\frac{\partial \theta(\infty, t)}{\partial x} = 0. \tag{10}$$

Taking the Laplace transform of equations (7)–(10) with respect to t and simplifying we get

$$\frac{d^2\bar{\theta}(x,s)}{dx^2} - \left(\frac{s}{\alpha}\right)\bar{\theta}(x,s) = -\frac{(1-R)}{\alpha\rho C_p}\bar{\dot{I}}_0(s)\mu\, e^{-\mu x} \tag{11}$$

$$\frac{d\bar{\theta}(0,s)}{dx} + \frac{h}{k}[(T_\infty - T_{int})/s - \bar{\theta}(0,s)] = 0 \tag{12}$$

$$\frac{d\bar{\theta}(\infty, s)}{dx} = 0. \tag{13}$$

The general solution of equation (11) can be expressed as

$$\bar{\theta}(x,s) = A\exp(-x\sqrt{(s/\alpha)}) + B\exp(-\mu x)\bar{\dot{I}}_0(s)/(s-b^2) \tag{14}$$

where

$$B = \mu(1-R)/\rho C_p \tag{15}$$

and

$$b^2 = \alpha\mu^2. \tag{16}$$

Note that the constant 'A' in equation (14) is evaluated by using the transformed boundary condition given by equation (12). This gives

$$A = \frac{A_1}{s(\sqrt{s}+a)} + \frac{A_2\bar{\dot{I}}_0(s)}{(s-b^2)(\sqrt{s}+a)} \tag{17}$$

where

$$A_1 = (h\sqrt{\alpha}/k)(T_\infty - T_{int}) \tag{18}$$

$$A_2 = -B\sqrt{\alpha}(\mu + h/k) \tag{19}$$

and

$$a = h\sqrt{\alpha}/k. \tag{20}$$

We can now express the transformed equation (14) by the following expression

$$\bar{\theta}(x,s) = \frac{A_1 \exp(-x\sqrt{(s/\alpha)})}{s(\sqrt{s}+a)} + \frac{A_2 \exp(-x\sqrt{(s/\alpha)})\bar{\dot{I}}_0(s)}{(s-b^2)(\sqrt{s}+a)} + \frac{B\exp(-\mu x)\bar{\dot{I}}_0(s)}{(s-b^2)} \tag{21}$$

which can be further simplified to

$$\bar{\theta}(x,s) = \frac{A_{10} \exp(-x\sqrt{(s/\alpha)})}{s(\sqrt{(s/\alpha)}+a_1)} + A_{20}\left[\frac{\exp(-x\sqrt{(s/\alpha)})}{s(\sqrt{(s/\alpha)}+a_1)} \times \frac{s\bar{\dot{I}}_0(s)}{(s-b^2)}\right] + \frac{B\exp(-\mu x)\bar{\dot{I}}_0(s)}{(s-b^2)} \tag{22}$$

where

$$A_{10} = A_1/\sqrt{\alpha} \tag{23}$$

$$A_{20} = A_2/\sqrt{\alpha} \tag{24}$$

$$a_1 = a/\sqrt{\alpha}. \tag{25}$$

Taking the inverse transform of equation (22) by using the transform relationships given in Appendix A, we find

$$\theta(x,t) = \frac{A_{10}}{a_1}\left[\operatorname{Erfc}\left(\frac{x}{2\sqrt{(\alpha t)}}\right) - E(a_1 x, \alpha t/x^2)\right] + \frac{A_{20}}{a_1}\left[\left\{\operatorname{Erfc}\left(\frac{x}{2\sqrt{(\alpha t)}}\right) - E(a_1 x, \alpha t/x^2)\right\} * \left\{\frac{\partial}{\partial t}[e^{b^2 t} * \dot{I}(t)]\right\}\right] + B\exp(-\mu x + b^2 t) * \dot{I}(t) \tag{26}$$

where '$*$' is the convolution with respect to t as defined by equation (A6).

Substituting the values of A_{10}, A_{20}, B, a_1 and b results in

$$\theta(x,t) = (T_\infty - T_{\text{int}})[\operatorname{Erfc}(x/2\sqrt{(\alpha t)} - E(hx/k, \alpha t/x^2)] - \frac{\mu(1-R)}{\rho C_p}\left[\frac{(\mu+h/k)}{(h/k)} \times \{\operatorname{Erfc}(x/2\sqrt{(\alpha t)}) - E(hx/k, \alpha t/x^2)\} * \left\{\frac{\partial}{\partial t}[e^{\alpha\mu^2 t} * \dot{I}(t)]\right\} - \{\exp(-\mu x + \alpha\mu^2 t)\} * \dot{I}(t)\right]. \tag{27}$$

It should be emphasized that equation (27) is the temperature solution due to the general time-dependent, spatially decaying laser source. This formulation may be used to discuss several heat-conduction problems arising in laser-induced processing of materials.

3. SOME CASES OF PRACTICAL INTEREST

In this section, we use the preceding formulation to discuss some particular heat conduction problems in which the radiation intensity varies with time.

3.1. *Instantaneous laser source*

We note that an instantaneous laser source of strength $\dot{I}_0(t) = I_0\delta(t)$, when substituted into equation (27) and using equation (A8) results in

$$\theta_1(x,t) = (T_\infty - T_{\text{int}})[\operatorname{Erfc}(x/2\sqrt{(\alpha t)}) - E(hx/k, \alpha t/x^2)] - \frac{I_0\mu(1-R)}{\rho C_p} \times \left[\frac{\alpha\mu^2(\mu+h/k)}{(h/k)}\{\operatorname{Erfc}(x/2\sqrt{(\alpha t)}) - E(hx/k, \alpha t/x^2)\} * e^{\alpha\mu^2 t} - \exp(-\mu x + \alpha\mu^2 t)\right]. \tag{28}$$

Using the properties of Laplace transform, Zubair and Chaudhry [5] have recently shown that the second term in the above equation can be reduced to

$$[\operatorname{Erfc}(x/2\sqrt{(\alpha t)}) - E(a_1 x, \alpha t/x^2)] * e^{b^2 t} = \frac{a_1}{2b^2(b/\sqrt{\alpha}+a_1)}\left[E(-bx/\sqrt{\alpha}, \alpha t/x^2) - \frac{(b/\sqrt{\alpha}+a_1)}{(b/\sqrt{\alpha}-a_1)}E(bx/\sqrt{\alpha}, \alpha t/x^2) + \frac{2a_1}{(b/\sqrt{\alpha}-a_1)}E(a_1 x, \alpha t/x^2)\right]. \tag{29}$$

Substituting into equation (28), we find after simplification that

$$\theta_1(x,t) = (T_\infty - T_{\text{int}})[\operatorname{Erfc}(x/2\sqrt{(\alpha t)}) - E(hx/k, \alpha t/x^2)] - \frac{I_0\mu(1-R)}{2\rho C_p} \times \left[E(-\mu x, \alpha t/x^2) - \frac{(\mu+h/k)}{(\mu-h/k)}E(\mu x, \alpha t/x^2) + \frac{2h/k}{(\mu-h/k)}E(hx/k, \alpha t/x^2) - 2\exp(-\mu x + \alpha\mu^2 t)\right]. \tag{30}$$

This is the same temperature solution as that discussed recently by Zubair and Chaudhry [5].

3.2. *Exponential-type laser source*

The exponential-type laser source of strength $\dot{I}_0(t) = \dot{I}_0 e^{\lambda^2 t}$, where λ^2 can be positive or negative, when substituted into equation (26) gives

$$\theta_2(x,t) = \frac{A_{10}}{a_1}[\mathrm{Erfc}(x/2\sqrt{(\alpha t)}) - E(a_1 x, \alpha t/x^2)] + \frac{A_{20}\dot{I}_0}{a_1}\left[\{\mathrm{Erfc}(x/2\sqrt{\alpha t}) - E(a_1 x, \alpha t/x^2)\} * \frac{\partial}{\partial t}(e^{b^2 t} * e^{\lambda^2 t})\right] + B\dot{I}_0\, e^{-\mu x}[e^{b^2 t} * e^{\lambda^2 t}]. \quad (31)$$

It should be noted that the second-term in the above equation can be simplified by using equation (A7) as

$$[\mathrm{Erfc}(x/2\sqrt{(\alpha t)}) - E(a_1 x, \alpha t/x^2)] * \left[\frac{\partial}{\partial t}(e^{b^2 t} * e^{\lambda^2 t})\right] = \frac{1}{(\lambda^2 - b^2)}[\{\mathrm{Erfc}(x/2\sqrt{(\alpha t)}) - E(a_1 x, \alpha t/x^2)\} * (\lambda^2 e^{\lambda^2 t} - b^2 e^{b^2 t})] \quad (32)$$

which can further be simplified by using equation (29) as

$$\frac{1}{(\lambda^2 - b^2)}[\{\mathrm{Erfc}(x/2\sqrt{(\alpha t)}) - E(a_1 x, \alpha t/x^2)\} * (\lambda^2 e^{\lambda^2 t} - b^2 e^{b^2 t})]$$
$$= \frac{1}{(\lambda^2 - b^2)}\left[\frac{a_1\lambda^2}{2\lambda^2(\lambda/\sqrt{\alpha} + a_1)} \times \left\{E(-\lambda x/\sqrt{\alpha}, \alpha t/x^2) - \frac{(\lambda/\sqrt{\alpha} + a_1)}{(\lambda/\sqrt{\alpha} - a_1)}E(\lambda x/\sqrt{\alpha}, \alpha t/x^2) + \frac{2a_1}{(\lambda/\sqrt{\alpha} - a_1)}E(a_1 x, \alpha t/x^2)\right\} - \frac{a_1 b^2}{2b^2(b/\sqrt{\alpha} + a_1)}\left\{E(-bx/\sqrt{\alpha}, \alpha t/x^2) - \frac{(b/\sqrt{\alpha} + a_1)}{(b/\sqrt{\alpha} - a_1)}E(bx/\sqrt{\alpha}, \alpha t/x^2) + \frac{2a_1}{(b/\sqrt{\alpha} - a_1)}E(a_1 x, \alpha t/x^2)\right\}\right], \quad (33)$$

Using the values of b, a_1, A_{10}, A_{20}, B and substituting equation (33) into equation (31), we get

$$\theta_2(x,t) = (T_\infty - T_{\mathrm{int}})[\mathrm{Erfc}(x/2\sqrt{(\alpha t)}) - E(hx/k, \alpha t/x^2)] - \frac{1}{(\lambda^2 - \alpha\mu^2)} \times \left[\frac{\mu(1-R)(\mu + h/k)\dot{I}_0}{2\rho C_p(\lambda/\sqrt{\alpha} + h/k)} \times \left\{E(-\lambda x/\sqrt{\alpha}, \alpha t/x^2) - \frac{(\lambda/\sqrt{\alpha} + h/k)}{(\lambda/\sqrt{\alpha} - h/k)}E(\lambda x/\sqrt{\alpha}, \alpha t/x^2) + \frac{2h/k}{(\lambda/\sqrt{\alpha} - h/k)}E(hx/k, \alpha t/x^2)\right\} - \frac{\mu(1-R)\dot{I}_0}{2\rho C_p}\left\{E(-\mu x, \alpha t/x^2) - \frac{(\mu + h/k)}{(\mu - h/k)}E(\mu x, \alpha t/x^2) + \frac{2h/k}{(\mu - h/k)}E(hx/k, \alpha t/x^2)\right\}\right] + \frac{\mu(1-R)\dot{I}_0\, e^{-\mu x}[e^{\lambda^2 t} - e^{\alpha\mu^2 t}]}{\rho C_p(\lambda^2 - \alpha\mu^2)}. \quad (34)$$

The temperature solution given by equation (34) may further be simplified by introducing dimensionless variables Θ_2, Fo, Bi, Bo, χ, τ as

$$\Theta_2 = \mathrm{Erfc}(1/2\sqrt{Fo}) - E(Bi, Fo) - Bo\left[\frac{(\chi + Bi)}{(\tau + Bi)}\left\{E(-\tau, Fo) - \frac{(\tau + Bi)}{(\tau - Bi)}E(\tau, Fo) + \frac{2Bi}{(\tau - Bi)}E(Bi, Fo)\right\} - E(-\chi, Fo) + \frac{(\chi + Bi)}{(\chi - Bi)}E(\chi, Fo) - \frac{2Bi}{(\chi - Bi)}E(Bi, Fo) - 2\exp(-\chi)\{\exp(\tau^2 Fo) - \exp(\chi^2 Fo)\}\right], \quad (35)$$

where

$$\Theta_2 = \theta_2(x,t)/(T_\infty - T_{\mathrm{int}}) = (T(x,t) - T_{\mathrm{int}})/(T_\infty - T_{\mathrm{int}}) \quad (36)$$

$$Fo = \alpha t/x^2 \quad (37)$$

$$Bi = hx/k \quad (38)$$

$$\tau = \lambda x/\sqrt{\alpha} \quad (39)$$

$$\chi = \mu x \quad (40)$$

and

$$Bo = B/2(T_\infty - T_{\mathrm{int}}) = \dot{I}_0\mu(1-R)/2\rho C_p(\lambda^2 - \alpha\mu^2)(T_\infty - T_{\mathrm{int}}). \quad (41)$$

We note that the first term in equation (35) is the solution for a semi-infinite body with a uniform initial temperature, T_{int}, no energy source, and a convective boundary condition; it accounts for a heat loss or gain when T_∞ and T_{int} are different. Even when $T_{\text{int}} = T_\infty$, there will still be a heat loss because the energy deposition is causing the surface temperature to rise. The term multiplied by Bo in equation (35) accounts for heat flow due to energy deposition.

It is of interest to calculate the heat flux at any 'x' by differentiating equation (34) with respect to x. We find that

$$
\begin{aligned}
q''_{x,2} &= -k\frac{\partial T}{\partial x} \\
&= h(T_\infty - T_{\text{int}})E(hx/k, \alpha t/x^2) \\
&\quad + \frac{\dot{I}_0\mu(1-R)k}{(\lambda^2-\alpha\mu^2)2\rho C_p}\left\{\frac{(\mu+h/k)}{(\lambda/\sqrt{\alpha}+h/k)}\right. \\
&\quad \times\left[\frac{-\lambda}{\sqrt{\alpha}}E(-\lambda x/\sqrt{\alpha}, \alpha t/x^2) - \frac{\exp(-x^2/4\alpha t)}{\sqrt{(\pi\alpha t)}}\right. \\
&\quad - \frac{(\lambda/\sqrt{\alpha}+h/k)}{(\lambda/\sqrt{\alpha}-h/k)}\left(\frac{\lambda}{\sqrt{\alpha}}E(\lambda x/\sqrt{\alpha}, \alpha t/x^2)\right. \\
&\quad \left. - \frac{\exp(-x^2/4\alpha t)}{\sqrt{(\pi\alpha t)}}\right) + \frac{2h/k}{(\lambda/\sqrt{\alpha}-h/k)} \\
&\quad \left.\times\left(\frac{h}{k}E(hx/k, \alpha t/x^2) - \frac{\exp(-x^2/4\alpha t)}{\sqrt{(\pi\alpha t)}}\right)\right] \\
&\quad + \mu E(-\mu x, \alpha t/x^2) + \frac{\exp(-x^2/4\alpha t)}{\sqrt{(\pi\alpha t)}} \\
&\quad + \frac{(\mu+h/k)}{(\mu-h/k)}\left(\mu E(\mu x, \alpha t/x^2) - \frac{\exp(-x^2/4\alpha t)}{\sqrt{(\pi\alpha t)}}\right) \\
&\quad \left. - \frac{2h/k}{(\mu-h/k)}\left(\frac{h}{k}E(hx/k, \alpha t/x^2) - \frac{\exp(-x^2/4\alpha t)}{\sqrt{(\pi\alpha t)}}\right)\right\} \\
&\quad + \frac{\mu^2 k(1-R)\dot{I}_0\,e^{-\mu x}[e^{\lambda^2 t} - e^{\alpha\mu^2 t}]}{\rho C_p(\lambda^2-\alpha\mu^2)}. \qquad (42)
\end{aligned}
$$

We can also express the heat flux in terms of dimensionless variables Q_2, Fo, Bi, Bo, τ and χ as

$$
\begin{aligned}
Q_2 &= E(Bi, Fo) + \frac{Bo}{Bi}\left[\left(\frac{\chi+Bi}{\tau+Bi}\right)\right. \\
&\quad \times\left\{-\tau E(-\tau, Fo) - \tau\left(\frac{\tau+Bi}{\tau-Bi}\right)\times E(\tau, Fo)\right. \\
&\quad \left. + \frac{2Bi^2}{(\tau-Bi)}E(Bi, Fo)\right\} - \chi E(-\chi, Fo) \\
&\quad + \chi\frac{(\chi+Bi)}{(\chi-Bi)}E(\chi, Fo) - \frac{2Bi^2}{(\chi-Bi)}E(Bi, Fo) \\
&\quad \left. + 2\chi\exp(-\chi)(\exp(\tau^2 Fo) - \exp(\chi^2 Fo))\right] \qquad (43)
\end{aligned}
$$

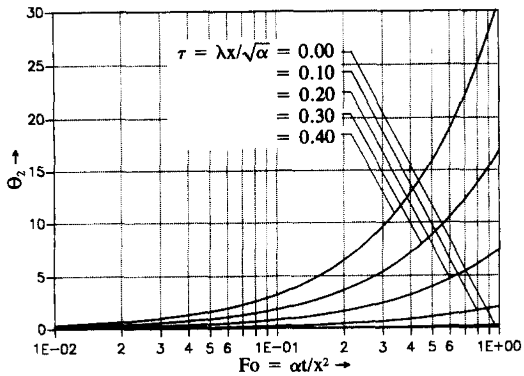

Fig. 1. Reduced temperature as a function of dimensionless time constant and Fourier number for an exponential-type laser source at $Bo = 100.0$, $Bi = 1.00$ and $\chi = 0.01$.

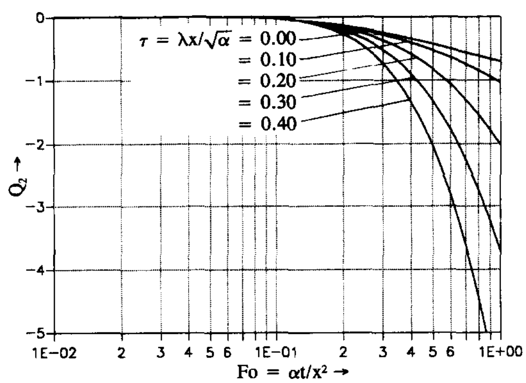

Fig. 2. Reduced heat flux as a function of dimensionless time constant and Fourier number for an exponential-type laser source at $Bo = 100.0$, $Bi = 1.00$ and $\chi = 0.01$.

where

$$Q_2 = q''_{x,2}/h(T_\infty - T_{\text{int}}) = -k\frac{\partial T}{\partial x}/h(T_\infty - T_{\text{int}}),$$

and other dimensionless parameters are described in equations (37)–(41).

The graphical representation of equations (35) and (43) is shown in Figs. 1 and 2, respectively. In these figures reduced temperature and heat flux solutions are presented as a function of the dimensionless time parameter Fo, for various values of reduced time constant τ. All the curves shown in these figures are drawn for the dimensionless energy absorption rate $Bo = 100.0$, Biot number $Bi = 1.00$ and reduced distance $\chi = 0.01$. It can be seen from Fig. 1 that the reduced temperature increases exponentially with Fo. On the other hand, the reduced heat flux (refer to Fig. 2) decreases with an increase in Fo. For example, at $\tau = 0.40$, there is about a ten-fold decrease in the reduced heat flux value when Fo is increased from 0.2 to 0.5. It should be noted that the particular case,

$\tau = 0$, in these figures represents the reduced temperature and heat flux solutions for the case of continuously operating laser source of constant strength, reported earlier by Blackwell [4].

3.2.1. *Wall temperature and heat flux.* The dimensionless wall temperature and heat flux can be determined by evaluating equations (35) and (43) 'at $x = 0$.' This gives $1/Fo = 0$, $Bi = 0$, $\chi = 0$ and $\tau = 0$. However, the products

$$\beta = Fo^{1/2}\chi = \sqrt{(\mu^2\alpha t)}$$

$$\eta = Fo^{1/2}Bi = h\sqrt{(\alpha t)}/k$$

$$\xi = Fo^{1/2}\tau = \lambda\sqrt{t}$$

$$\phi = \chi/Bi = \mu k/h$$

$$\Gamma = \tau/Bi = \lambda k/h\sqrt{\alpha}$$

remain finite, because the geometric distance 'x' has been suppressed in these. For this reason, the dimensionless temperature and heat flux at the wall is given by the following simplified equations:

$$\Theta_{21} = 1 - \exp(\eta^2)\,\mathrm{Erfc}(\eta) - Bo\left[\left(\frac{\phi+1}{\Gamma+1}\right)\left\{\exp(\xi^2)\,\mathrm{Erfc}(-\xi) - \left(\frac{\Gamma+1}{\Gamma-1}\right)\exp(\xi^2)\,\mathrm{Erfc}(\xi) + \left(\frac{2}{\Gamma-1}\right)\exp(\eta^2)\,\mathrm{Erfc}(\eta)\right\} - \exp(\beta^2)\,\mathrm{Erfc}(-\beta) + \left(\frac{\phi+1}{\phi-1}\right)\exp(\beta^2)\,\mathrm{Erfc}(\beta) - \left(\frac{2}{\phi-1}\right)\exp(\eta^2)\,\mathrm{Erfc}(\eta) - 2[\exp(\xi^2) - \exp(\beta^2)]\right] \quad (44)$$

and

$$Q_{21} = \exp(\eta^2)\,\mathrm{Erfc}(\eta) - Bo\left[\left(\frac{\phi+1}{\Gamma+1}\right) \times \left\{-\Gamma\exp(\xi^2)\,\mathrm{Erfc}(-\xi) - \Gamma\left(\frac{\Gamma+1}{\Gamma-1}\right) \times \exp(\xi^2)\,\mathrm{Erfc}(\xi) + \left(\frac{2}{\Gamma-1}\right)\exp(\eta^2)\,\mathrm{Erfc}(\eta)\right\} - \phi\exp(\beta^2)\,\mathrm{Erfc}(-\beta) + \phi\left(\frac{\phi+1}{\phi-1}\right) \times \exp(\beta^2)\,\mathrm{Erfc}(\beta) - \left(\frac{2}{\Gamma-1}\right)\exp(\eta^2)\,\mathrm{Erfc}(\eta) + 2\phi[\exp(\xi^2) - \exp(\beta^2)]\right]. \quad (45)$$

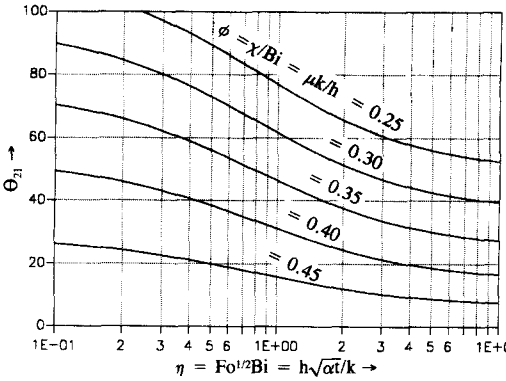

Fig. 3. Reduced wall temperature as a function of dimensionless parameter η and ϕ at $Bo = 100.00$, $\beta = 1.00$, $\xi = 1.00$ and $\Gamma = 0.50$.

Figures 3 and 4 represent the dimensionless temperature and heat flux at the wall in terms of dimensionless parameter η and ϕ at $Bo = 100.0$, $\beta = 1.00$, $\xi = 1.00$ and $\Gamma = 0.50$. We note that the reduced temperature plots (refer to Fig. 3) are represented by characteristic Gaussian-type curves, where the temperature decreases with an increase in values of η and ϕ. As expected, the reduced heat flux plots (refer to Fig. 4) show that there is an increase in the reduced flux with η and ϕ. For example, at $\eta = 1.00$, there is about 50% increase in Q_{21} when ϕ is increased from 0.25 to 0.40. It should be noted that for a given material, low values of η and ϕ imply a small time and high convection coefficient.

3.2.2. *Laser source of constant strength.* We note that $\lambda^2 = 0$ (or $\tau = 0$) in equations (35) and (43) reduces to the case of continuously operating laser source of constant strength $\dot{I}_0(t) = \dot{I}_0$. This gives

$$\Theta_{22} = \mathrm{Erfc}(1/2\sqrt{Fo}) - E(Bi, Fo)$$

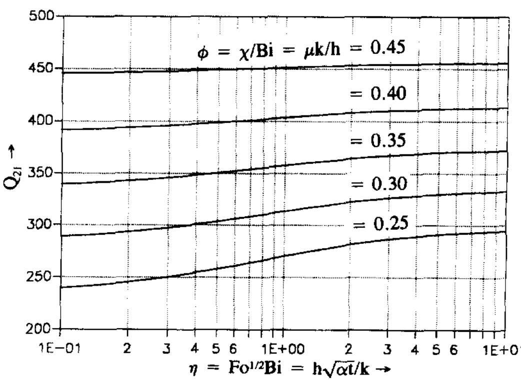

Fig. 4. Reduced wall heat flux as a function of dimensionless parameter η and ϕ at $Bo = 100.00$, $\beta = 1.00$, $\xi = 1.00$ and $\Gamma = 0.50$.

$$- \mathrm{Bo}\Big[(\chi/\mathrm{Bi}+1)\{2\,\mathrm{Erfc}(1/2\sqrt{Fo}) - 2E(Bi,Fo)\} - E(-\chi,Fo) + \left(\frac{\chi+Bi}{\chi-Bi}\right) \times E(\chi,Fo) - \frac{2Bi}{(\chi-Bi)}E(Bi,Fo) - 2\exp(-\chi)(1-\exp(\chi^2 Fo))\Big] \quad (46)$$

and the reduced heat flux is

$$Q_{22} = E(Bi,Fo) - \frac{Bo}{Bi}\Big[2Bi(\chi/Bi+1) \times E(Bi,Fo) + \chi E(-\chi,Fo) - \chi\left(\frac{\chi+Bi}{\chi-Bi}\right) \times E(\chi,Fo) + \frac{2Bi^2}{(\chi-Bi)}E(Bi,Fo) - 2\chi\exp(-\chi)(1-\exp(\chi^2 Fo))\Big] \quad (47)$$

which are the same temperature and heat flux solutions as those discussed by Blackwell [4]. It should, however, be emphasized that with the introduction of the function E, the solutions are presented in a more compact form than those by Blackwell.

3.2.3. *No heat generation.* We note that for the case of no heat generation in the solid, the reduced temperature and heat flux can be determined by evaluating equations (35) and (43) at $\chi = 0$, $Bo = 0$ and $\tau = 0$. This gives

$$\Theta_{23} = \mathrm{Erfc}(1/2\sqrt{Fo}) - E(Bi,Fo) \quad (48)$$

and

$$Q_{23} = E(Bi,Fo) \quad (49)$$

where the function E is defined in the appendix as equation (A4). We note that this is the same temperature solution as that reported by Carslaw and Jaeger [6] and Grigull and Sandner [7].

3.2.4. *Constant surface temperature.* Another important solution which can be recovered from the present analysis is for the case of constant surface temperature, that is, substituting $1/h = 0$ and $T_{\mathrm{int}} = T_\infty$ in equations (34) and (42). We find

$$\Theta_3 = -E(-\tau,Fo) - E(\tau,Fo) + E(-\chi,Fo) + E(\chi,Fo) + 2\exp(-\chi)[\exp(\tau^2 Fo) - \exp(\chi^2 Fo)] \quad (50)$$

$$Q_3 = \frac{1}{\chi}[-\tau E(-\tau,Fo) + \tau E(\tau,Fo) + \chi E(-\chi,Fo) - \chi E(\chi,Fo)] + 2\exp(-\chi)(\exp(\tau^2 Fo) - \exp(\chi^2 Fo)) \quad (51)$$

where

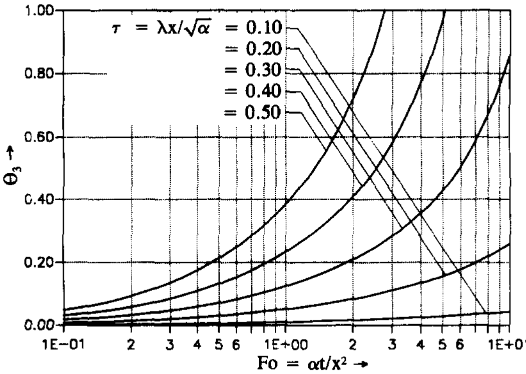

Fig. 5. Reduced temperature as a function of dimensionless time constant and Fourier number for the case of constant surface temperature at $\chi = 0.05$.

$$\Theta_3 = \frac{2\rho C_p(\lambda^2 - \alpha\mu^2)\theta(x,t)}{\dot{I}_0\mu(1-R)} = \frac{2\rho C_p(\lambda^2 - \alpha\mu^2)(T(x,t) - T_{\mathrm{int}})}{\dot{I}_0\mu(1-R)} \quad (52)$$

and

$$Q_3 = \frac{2\rho C_p(\lambda^2 - \alpha\mu^2)q''_{x,2}}{\dot{I}_0\mu^2 k(1-R)} = \frac{2\rho C_p(\lambda^2 - \alpha\mu^2)\left(-k\frac{\partial T}{\partial x}\right)}{\dot{I}_0\mu^2 k(1-R)}. \quad (53)$$

The graphical representation for the case of constant surface temperature solution given by the above equations, is shown in Figs. 5 and 6. In these figures, the reduced temperature and heat flux are presented in terms of Fo and τ at the reduced distance $\chi = 0.05$. We note that the reduced temperature increases exponentially with the Fourier number, whereas the heat flux decreases with an increase in Fo. For example, at $\tau = 0.50$, there is approximately a seven-fold increase in the reduced temperature value when Fo is varied from 0.1 to 1. It should be noted that for a fixed spatial

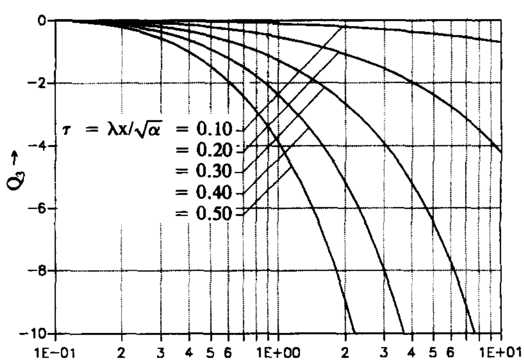

Fig. 6. Reduced heat flux as a function of dimensionless time constant and Fourier number for the case of constant surface temperature at $\chi = 0.05$.

location in a given material, an increase in Fo implies that there is also an increase in the time of laser exposure. On the other hand, Fig. 6 shows that the reduced heat flux decreases with an increase in Fo and τ. For example, at $\tau = 0.50$, the flux decreases from 0 to -2 when Fo is increased from 0.10 to 0.60, respectively.

4. CONCLUDING REMARKS

The closed-form temperature and heat flux solutions for a time-dependent laser source when subjected to convective-boundary conditions, are presented in a compact form by using the generalized representation of an error function. The solutions are discussed for the case of laser source of the form $\dot{I}_0(t) = \dot{I}_0 e^{\lambda^2 t}$, where λ^2 can be positive or negative or equal to zero. All the results are presented in terms of the reduced time (Fo), reduced energy absorption rate (Bo), Biot number (Bi), dimensionless distance (χ), and reduced time constant (τ). Additional dimensionless parameters are also identified for discussing the wall temperature and heat flux solutions. As expected, the graphical representation of the solutions indicates a strong dependence of Fo, both for the case of convective cooling of the exposed surface as well as that of constant surface temperature cases.

The contributions due to uniform initial temperature, no energy source, and a convective cooling of the exposed surface, as well as that due to heat flow caused by energy deposition, are identified in the solutions. In addition, solutions of special cases which include (i) instantaneous laser source and (ii) laser source of constant strength discussed in the literature are recovered from the present formulation of a general time-dependent exposure of laser radiation which is absorbed within the material and has the effect of an internally distributed heat source.

It should also be noted that the analysis discussed in this paper is limited for convective cooling of the exposed material surface; however, radiation losses will be important for most materials when the surface temperature approaches phase-change temperatures. We emphasize that for applications in which radiation losses and an ablating boundary conditions are important, a numerical model is essential that may be checked under the limiting conditions against the analytical solutions discussed in this paper.

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APPENDIX A

The following relationships are useful in finding the inverse transform of equation (22):

$$\mathscr{L}^{-1}\left[\frac{1}{(s-\alpha)}\right] = e^{\alpha t} \tag{A1}$$

$$\mathscr{L}^{-1}\left[\frac{\exp(-x\sqrt{(s/\alpha)})}{s(\sqrt{(s/\alpha)}+c)}\right] = \frac{1}{c}\left[\operatorname{Erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - E(cx, \alpha t/x^2)\right] \tag{A2}$$

$$\mathscr{L}^{-1}\left[\frac{\exp(-x\sqrt{(s/\alpha)})}{(\sqrt{(s/\alpha)}+c)}\right] = \left(\frac{\alpha}{\pi t}\right)^{1/2}\exp(-x^2/4\alpha t) - \alpha c E(cx, \alpha t/x^2)\,(x > 0, \alpha > 0) \tag{A3}$$

where the function $E(cx, \alpha t/x^2)$ is given by [9]

$$E(cx, \alpha t/x^2) = \exp[cx + c^2x^2(\alpha t/x^2)] \times \operatorname{Erfc}\left(\frac{x}{2\sqrt{(\alpha t)}} + cx\sqrt{(\alpha t/x^2)}\right) \tag{A4}$$

and the differentiation with respect to 'x' is

$$\frac{\partial}{\partial x}[E(cx, \alpha t/x^2)] = cE(cx, \alpha t/x^2) - \frac{1}{\sqrt{(\pi\alpha t)}}\exp(-x^2/4\alpha t). \tag{A5}$$

The convolution of the functions $f(t)$ and $g(t)$ is given by

$$\{f(t)\} * \{g(t)\} = \int_0^t f(t-u)g(u)\,du. \tag{A6}$$

In particular,

$$e^{at} * e^{ct} = \frac{1}{(c-a)}[e^{ct} - e^{at}] \tag{A7}$$

and

$$f(t) * \delta(t) = f(t). \tag{A8}$$